

# $L^\infty$ SOLUTIONS FOR A MODEL OF POLYTROPIC GAS FLOW WITH DIFFUSIVE ENTROPY

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ABSTRACT. We establish the global existence of  $L^\infty$  solutions for a model of polytropic gas flow with diffusive entropy. The result is obtained by showing the convergence of a class of finite difference schemes, which includes the Lax–Friedrichs and Godunov schemes. Such convergence is achieved by proving the estimates required for the application of the compensated compactness theory.

## 1. INTRODUCTION

We consider the following system modeling isentropic gas flow with smoothly varying entropy. The model reads in Eulerian coordinates

$$(1.1) \quad \rho_t + m_x = 0,$$

$$(1.2) \quad m_t + \left(\frac{m^2}{\rho} + p(\rho, S)\right)_x = 0,$$

$$(1.3) \quad (\rho S)_t + (mS)_x = \left(\frac{1}{\rho} S_x\right)_x,$$

where

$$p(\rho, S) = \kappa e^{(\gamma-1)S/\mathfrak{R}} \rho^\gamma,$$

where  $\mathfrak{R} > 0$  and  $\gamma > 1$  are constants, and  $\kappa = \frac{1}{4\gamma}(\gamma - 1)^2$ . Here  $\rho$  represents the gas density,  $m$  is the momentum defined as  $m = \rho u$ , where  $u$  is the gas velocity,  $p$  represents the gas pressure, and  $S$  stands for the entropy. The system (1.1)–(1.3) is a mathematical model intended to approximate the more physical model where equation (1.3) is replaced by the energy conservation law, which for smooth solutions is equivalent to the equation  $(\rho S)_t + (mS)_x = 0$ , and this motivates our mathematical model.

Initial data are given by

$$(1.4) \quad \rho(x, 0) = \rho_0(x), \quad m(x, 0) = m_0(x),$$

$$(1.5) \quad S(x, 0) = S_0(x) = \sigma(y_0(x)), \quad y_0(x) = \int_0^x \rho_0(z) dz.$$

Assume that

$$(1.6) \quad \rho_0, m_0, \frac{m_0}{\rho_0} \in L^\infty(\mathbb{R}), \quad \rho_0 \geq 0, \quad \sigma \in W_{\text{loc}}^{3,2}(\mathbb{R}).$$

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In particular, the initial data (and the solution) allows for the occurrence of vacuum. In addition, we also assume that  $\sigma$  is periodic with period, say,  $2\pi$ , that is,

$$(1.7) \quad \sigma(y + 2\pi) = \sigma(y), \quad y \in \mathbb{R}.$$

We remark that assumption (1.6), imposed on  $\sigma$ , implies that the solution of the heat equation with initial data  $\sigma$ ,

$$(1.8) \quad \tilde{\sigma}(y, t) := \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-(y-z)^2/4t} \sigma(z) dz,$$

satisfies

$$(1.9) \quad |\tilde{\sigma}(y, t) - \bar{\sigma}|, |\tilde{\sigma}_y(y, t)|, |\tilde{\sigma}_{yy}(y, t)| \leq C_0 e^{-t}, \quad \text{with} \quad \bar{\sigma} := \frac{1}{2\pi} \int_0^{2\pi} \sigma(z) dz,$$

for some absolute constant  $C_0 > 0$ . Indeed, (1.6) and (1.7) imply the absolute convergence of the Fourier series of  $\sigma$ ,  $\sigma'$  and  $\sigma''$ . On the other hand, a straightforward calculation shows that

$$\begin{aligned} \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-(y-z)^2/4t} e^{ikz} dz &= \frac{e^{(-y^2 + (y+2ikt)^2)/4t}}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{(z-(y+2ikt))^2}{4t}} dz \\ &= e^{iky - k^2 t}, \end{aligned}$$

for any  $k \in \mathbb{R}$ , which then gives the asserted asymptotic behavior, by plugging the Fourier series for  $\sigma$ ,  $\sigma'$  and  $\sigma''$  in (1.8) and the corresponding equations for  $\tilde{\sigma}_y$  and  $\tilde{\sigma}_{yy}$ , obtained from (1.8) by replacing  $\sigma$  by  $\sigma'$  and  $\sigma''$ , respectively.

We have the following definition of weak solution.

**Definition 1.1.** We say that a function  $(\rho, m, S) \in L^\infty(\mathbb{R} \times (0, \infty))$  is a weak solution to (1.1)–(1.5) if:

- (i)  $m/\rho \in L^\infty(\mathbb{R} \times (0, \infty))$ ;
- (ii) for all  $\phi \in C_0^\infty(\mathbb{R}^2)$ ,

$$(1.10) \quad \int_{\mathbb{R} \times (0, \infty)} (\rho, m)(x, t) \phi_t + \left(m, \frac{m^2}{\rho} + p(S, \rho)\right)(x, t) \phi_x dx dt + \int_{\mathbb{R}} (\rho_0, m_0)(x) \phi(x, 0) dx = 0;$$

(iii)

$$(1.11) \quad S(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(y(x, t) - z)^2}{4t}} \sigma(z) dz,$$

where

$$(1.12) \quad y(x, t) = \int_0^x \rho(z, t) dz - \int_0^t m(0, s) ds.$$

We observe that away from vacuum, equation (1.3), through the Lagrange transformation,  $(x, t) \mapsto (y(x, t), t)$ , with  $y(x, t)$  given by (1.12), becomes

$$S_t = S_{yy},$$

and this justifies (iii) of Definition 1.1.

Indeed, the interplay between the Lagrangian and Eulerian formulation of the model is important. For the record we note that the model (1.1)–(1.3) reads in Lagrangian coordinates

$$(1.13) \quad \begin{aligned} v_t - u_y &= 0, \\ u_t + p(v, S)_y &= 0, \\ S_t &= S_{yy} \end{aligned}$$

where  $v = 1/\rho$  is the specific volume. We remark that, despite the fact that system (1.13) has a form much simpler than (1.1), (1.2), (1.3), the possibility of occurrence of vacuum turns the direct analysis of the Cauchy problem for (1.13) a very difficult task and so, as in the isentropic case, a better strategy is to proceed with the analysis of the corresponding problem in Eulerian coordinates, that is, (1.1)–(1.5).

Our main result reads as follows.

**Theorem 1.1.** *There exists a constant  $r(\gamma) > 0$  such that if  $\|(\rho_0, m_0)\|_\infty < r(\gamma)$ , then there exists a global weak solution to the Cauchy problem (1.1)–(1.5) satisfying an entropy inequality of the form*

$$(1.14) \quad \eta_*(\rho, m, S)_t + q_*(\rho, m, S)_x \leq -Ce^{-t},$$

in the sense of distributions, for some  $C > 0$  depending on  $L^\infty$  bounds for  $\rho, m, S$ , where

$$(1.15) \quad \eta_*(\rho, m, S) = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}e^{(\gamma-1)S/\Re}\rho^\gamma, \quad q_*(\rho, m, S) = u\eta_*(\rho, m, S) + pu.$$

Moreover,  $r(\gamma) \rightarrow \infty$  as  $\gamma \rightarrow 1+$ . Further, if  $\rho_0, m_0$  are periodic with period  $L$  such that  $y_0(L) = 2\pi$ , we have the following decay

$$(1.16) \quad \lim_{t \rightarrow \infty} \int_0^L |(\rho(x, t), m(x, t), S(x, t)) - (\bar{\rho}, \bar{m}, \bar{S})| dx = 0,$$

where  $\bar{\rho}, \bar{m}, \bar{S}$  are the mean values of  $\rho_0, m_0, S_0$ , respectively.

## 2. BACKGROUND RESULTS

Let us first recall results for the  $p$ -system for a polytropic gas in Eulerian coordinates. More precisely, we consider the system

$$(2.1) \quad \rho_t + m_x = 0,$$

$$(2.2) \quad m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = 0,$$

where the pressure is given by  $p(\rho) = \kappa e^{(\gamma-1)S/\Re}\rho^\gamma$ . For later use we observe that we can rewrite the conserved quantities in terms of the other variables, viz.,

$$(2.3) \quad \rho = \rho(p, S) = \left(\frac{p}{\kappa}\right)^{1/\gamma} e^{(\gamma-1)S/(\gamma\Re)}, \quad m = m(u, p, S) = \rho u = u \left(\frac{p}{\kappa}\right)^{1/\gamma} e^{(\gamma-1)S/(\gamma\Re)}.$$

Here we consider the isentropic case where the entropy  $S$  is considered a constant. Recall that the functions

$$(2.4) \quad w = u + \frac{1}{\theta}(p_\rho)^{1/2} = u + e^{\theta S/\Re}\rho^\theta = u + \left(\frac{p}{\kappa}\right)^{\theta/\gamma} e^{-\theta S/(\gamma\Re)},$$

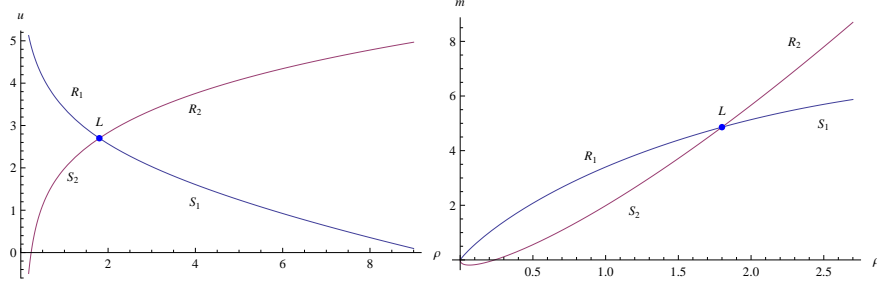
$$(2.5) \quad z = u - \frac{1}{\theta}(p_\rho)^{1/2} = u - e^{\theta S/\Re}\rho^\theta = u - \left(\frac{p}{\kappa}\right)^{\theta/\gamma} e^{-\theta S/(\gamma\Re)},$$

with  $\theta = \frac{1}{2}(\gamma - 1)$ , form a pair of Riemann invariants for system (2.1)–(2.2) in the isentropic case where  $S$  is constant. A standard calculation (see, e.g., [6, 3]) yields that the rarefaction curves are given by

$$m = \frac{m_l}{\rho_l}\rho \pm \gamma^{1/2}e^{\theta S/\Re}\rho(\rho^\theta - \rho_l^\theta),$$

while the Hugoniot locus reads

$$m = \frac{m_l}{\rho_l}\rho \pm \theta e^{\theta S/\Re}\rho \left(\frac{1}{\rho\rho_l}(\rho^\gamma - \rho_l^\gamma)(\rho - \rho_l)\right)^{1/2},$$

FIGURE 1. The wave curves for the  $p$ -system for a given left state.

from a given left state  $(\rho_l, m_l)$ . When we involve the entropy condition we find that the wave curves equal

$$(2.6) \quad W_1(\rho_l, m_l) : \quad m = \frac{m_l}{\rho_l} \rho - \begin{cases} \gamma^{1/2} e^{\theta S/\Re} \rho (\rho^\theta - \rho_l^\theta) & \text{for } \rho \leq \rho_l, \\ \theta e^{\theta S/\Re} \rho \left( \frac{1}{\rho_l} (\rho^\gamma - \rho_l^\gamma) (\rho - \rho_l) \right)^{1/2} & \text{for } \rho \geq \rho_l, \end{cases}$$

$$(2.7) \quad W_2(\rho_l, m_l) : \quad m = \frac{m_l}{\rho_l} \rho + \begin{cases} \theta e^{\theta S/\Re} \rho \left( \frac{1}{\rho_l} (\rho^\gamma - \rho_l^\gamma) (\rho - \rho_l) \right)^{1/2} & \text{for } \rho \leq \rho_l, \\ \gamma^{1/2} e^{\theta S/\Re} \rho (\rho^\theta - \rho_l^\theta) & \text{for } \rho \geq \rho_l. \end{cases}$$

In the variables  $(\rho, u)$  we find

$$(2.8) \quad W_1(\rho_l, u_l) : \quad u = u_l - \begin{cases} \gamma^{1/2} e^{\theta S/\Re} (\rho^\theta - \rho_l^\theta) & \text{for } \rho \leq \rho_l, \\ \theta e^{\theta S/\Re} \left( \frac{1}{\rho_l} (\rho^\gamma - \rho_l^\gamma) (\rho - \rho_l) \right)^{1/2} & \text{for } \rho \geq \rho_l, \end{cases}$$

$$(2.9) \quad W_2(\rho_l, u_l) : \quad u = u_l + \begin{cases} \theta e^{\theta S/\Re} \left( \frac{1}{\rho_l} (\rho^\gamma - \rho_l^\gamma) (\rho - \rho_l) \right)^{1/2} & \text{for } \rho \leq \rho_l, \\ \gamma^{1/2} e^{\theta S/\Re} (\rho^\theta - \rho_l^\theta) & \text{for } \rho \geq \rho_l. \end{cases}$$

An important property of the  $p$ -system is that the Riemann invariants provide invariant regions. More specifically, (see, e.g., [3, Lemma 5]) if  $(\rho_0(x), m_0(x)) \in \Omega = \{(\rho, m) \mid w \leq w_0, z \geq z_0, w - z \geq 0\}$  for all  $x \in \mathbb{R}$ , then also the solution  $(\rho(x, t), m(x, t))$  will remain in  $\Omega$ , that is,  $(\rho(x, t), m(x, t)) \in \Omega$  for  $(x, t) \in \mathbb{R} \times [0, \infty)$ .

An entropy-entropy flux pair  $(\eta, q)$  for the  $p$ -system satisfies for smooth solutions

$$\eta(\rho, m)_t + q(\rho, m)_x = 0.$$

Consistency with the system (2.1)–(2.2) requires

$$(2.10) \quad \nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla F(\rho, m),$$

where  $F = (m, \frac{m^2}{\rho} + p(\rho))$  is the flux function of the  $p$ -system. A particular choice of entropy-entropy flux pair  $(\eta_*, q_*)$  reads

$$(2.11) \quad \eta_* = \frac{m^2}{2\rho} + \frac{p}{\gamma - 1} = \frac{1}{2} \rho u^2 + \frac{\kappa}{\gamma - 1} e^{(\gamma-1)S/R} \rho^\gamma,$$

$$(2.12) \quad q_* = u\eta_* + pu = u\eta_* + u\kappa e^{(\gamma-1)S/R} \rho^\gamma.$$

More generally, the weak entropy-entropy flux pairs  $(\eta, q)$  constitute a class of entropy-entropy flux pairs of particular interest in isentropic gas dynamics, as first

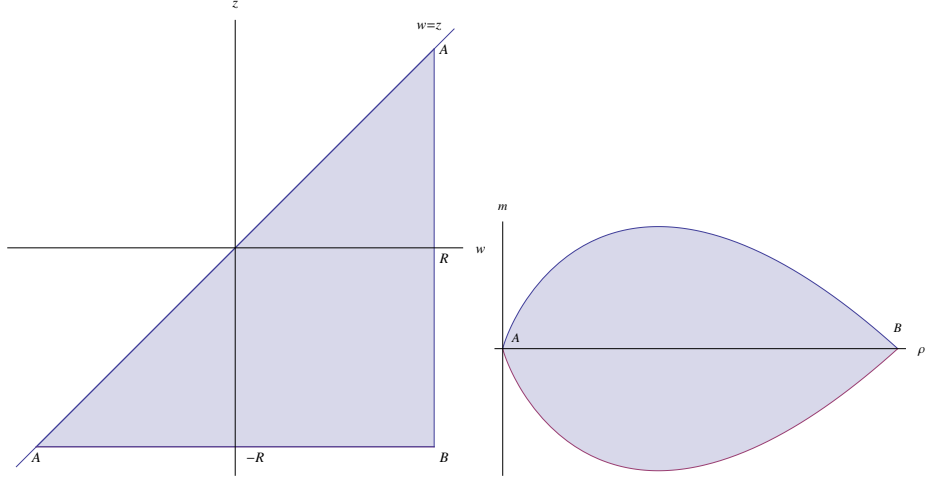


FIGURE 2. The triangle in the Riemann invariants (left) is mapped into the indicated region bounded by rarefaction waves in the  $(\rho, m)$ -plane.

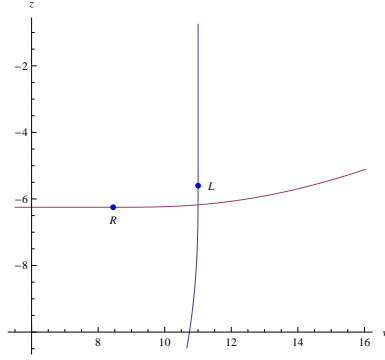


FIGURE 3. The solution of the Riemann problem using the Riemann invariants as coordinates. Through the right state backward Riemann invariants are drawn.

pointed out in [5], and they are characterized by the following conditions at the vacuum line:

$$\eta(\rho, u)|_{\rho=0} = 0, \quad \eta_\rho(\rho, u)|_{\rho=0} = g(u),$$

for some continuous function  $g$ . Let us denote

$$\chi(\rho, u; S) = \left(\frac{p}{\rho} - u^2\right)_+^\lambda,$$

where  $(a)_+ = \max\{0, a\}$  and  $\lambda = \frac{3-\gamma}{2(\gamma-1)}$ . As observed in [8], weak entropy-entropy flux pairs can be given by the integral formulas

$$(2.13) \quad \eta(\rho, u) = \int_{\mathbb{R}} g(\xi) \chi(\rho, \xi - u) d\xi,$$

$$(2.14) \quad q(\rho, u) = \int_{\mathbb{R}} g(\xi) (\theta \xi + (1 - \theta)u) \chi(\rho, \xi - u) d\xi.$$

*Remark 2.1.* Observe that the entropy pair  $(\eta_*, q_*)$ , defined in (1.15), is a weak convex entropy pair. Moreover, for any weak entropy pair  $(\eta, q)$  there exists a constant  $C_\eta > 0$  such that  $\eta + C_\eta \eta_*$  is convex.

Let us now turn to the full system

$$(2.15) \quad \rho_t + m_x = 0,$$

$$(2.16) \quad m_t + \left(\frac{m^2}{\rho} + p(\rho, S)\right)_x = 0,$$

$$(2.17) \quad (\rho S)_t + (mS)_x = 0,$$

where the pressure  $p$  is given as above. The Riemann problem is the initial value problem for the system (2.15)–(2.17) with special initial data consisting of a single jump between two constant states, viz.

$$\left. \begin{pmatrix} \rho \\ m \\ \rho S \end{pmatrix} \right|_{t=0} (x) = \begin{cases} \begin{pmatrix} \rho_l \\ m_l \\ \rho_l S_l \end{pmatrix} & \text{for } x < 0, \\ \begin{pmatrix} \rho_r \\ m_r \\ \rho_r S_r \end{pmatrix} & \text{for } x > 0. \end{cases}$$

The system (2.15)–(2.17) possesses three eigenfields associated with the eigenvalues

$$\lambda_1 = u - \sqrt{p_\rho}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{p_\rho}.$$

The solution to a Riemann problem for system (2.15)–(2.17) may be described using the coordinates  $w, z, S$ , that is, the Riemann invariants for the  $p$ -system and the entropy, in the following way. Consider first the case when the solution does not contain vacuum. The solution of the Riemann problem, starting from the left state  $(\rho_l, m_l, S_l)$ , consists of a slow wave in which the entropy  $S$  remains constant (i.e., in the  $(w, z)$ -plane determined by  $S = S_l$ ), followed by a contact discontinuity in which the velocity  $u$  and the pressure  $p$  remain unchanged, and finally a fast wave with constant entropy  $S$  (i.e., in the  $(w, z)$ -plane determined by  $S = S_r$ ) connected with the given right state  $(\rho_r, m_r, S_r)$ . Along the slow wave we can write the Riemann invariants as<sup>1</sup>

$$(2.18) \quad \begin{aligned} w &= u_1(\rho; \rho_l, u_l, S_l) + e^{\theta S_l / \Re} \rho^\theta, \\ z &= u_1(\rho; \rho_l, u_l, S_l) - e^{\theta S_l / \Re} \rho^\theta, \end{aligned}$$

where  $u = u_1(\rho; \rho_l, u_l, S_l)$  is the slow wave given by (2.6). For the fast wave we consider the *backward* wave (i.e., consisting of the states that can be connected to a given right state from the left), and the Riemann invariants read

$$(2.19) \quad \begin{aligned} w &= \tilde{u}_2(\rho; \rho_r, u_r, S_r) + e^{\theta S_r / \Re} \rho^\theta, \\ z &= \tilde{u}_2(\rho; \rho_r, u_r, S_r) - e^{\theta S_r / \Re} \rho^\theta, \end{aligned}$$

where  $u = \tilde{u}_2(\rho; \rho_r, u_r, S_r)$  is the fast *backward* wave corresponding to (2.7). The contact discontinuity, with pressure  $p^*$  and velocity  $u^*$ , jumps from a left density  $\rho_l^*$  to a right density  $\rho_r^*$  determined by

$$(2.20) \quad \begin{aligned} p^* &= \kappa e^{(\gamma-1)S_l / \Re} (\rho_l^*)^\gamma = \kappa e^{(\gamma-1)S_r / \Re} (\rho_r^*)^\gamma, \\ u^* &= u_1(\rho_l^*; \rho_l, u_l, S_l) = \tilde{u}_2(\rho_r^*; \rho_r, u_r, S_r), \end{aligned}$$

which yields

$$(2.21) \quad \frac{\rho_l^*}{\rho_r^*} = e^{(\gamma-1)(S_r - S_l) / (\gamma \Re)}$$

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<sup>1</sup>It turns out to be easier to describe the solution using the speed  $u$  rather than the momentum  $m$  as a variable.

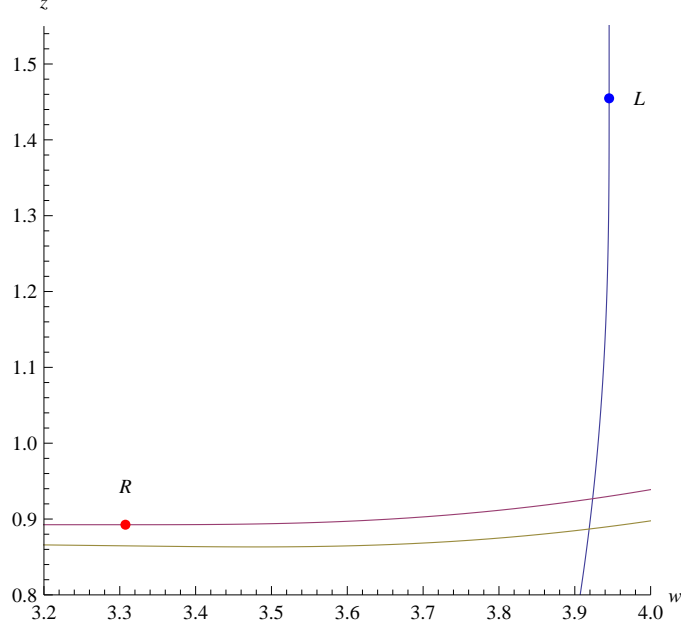


FIGURE 4. The slow Riemann invariant through the left state (blue curve), and the backward fast Riemann invariant through the right state (red curve). In addition the yellow curve  $(\bar{w}, \bar{z})$ , whose intersection with the slow Riemann invariant determines the contact discontinuity.

to be inserted in the second equation for the velocity,  $u_1 = \tilde{u}_2$ , to determine  $\rho_l^*$  and  $\rho_r^*$ . In terms of the Riemann invariants we find that  $w$  jumps from  $u^* + (p^*/\kappa)^{\theta/\gamma} e^{-\theta S_l/(\gamma \mathfrak{R})}$  to  $u^* + (p^*/\kappa)^{\theta/\gamma} e^{-\theta S_r/(\gamma \mathfrak{R})}$ , and similarly  $z$  jumps from  $u^* - (p^*/\kappa)^{\theta/\gamma} e^{-\theta S_l/(\gamma \mathfrak{R})}$  to  $u^* - (p^*/\kappa)^{\theta/\gamma} e^{-\theta S_r/(\gamma \mathfrak{R})}$ . An alternative way to describe the contact discontinuity is the following. Consider a point on the backward fast wave curve with Riemann invariants  $(w, z)$  given by (2.19), which we can write as  $w = \tilde{u}_2 + (p/\kappa)^{\theta/\gamma} e^{-\theta S_r/(\gamma \mathfrak{R})}$  and  $z = \tilde{u}_2 - (p/\kappa)^{\theta/\gamma} e^{-\theta S_r/(\gamma \mathfrak{R})}$ . Construct now another curve  $(\bar{w}, \bar{z})$ , given as a Riemann invariant with the same velocity  $\tilde{u}_2$  and pressure  $p$  as  $(w, z)$ , but with the entropy  $S_r$  replaced by  $S_l$ , that is,

$$\bar{w} = \tilde{u}_2 + (p/\kappa)^{\theta/\gamma} e^{-\theta S_l/(\gamma \mathfrak{R})}, \quad \bar{z} = \tilde{u}_2 - (p/\kappa)^{\theta/\gamma} e^{-\theta S_l/(\gamma \mathfrak{R})}.$$

We find

$$\begin{aligned} w + z &= 2\tilde{u}_2 = \bar{w} + \bar{z}, \\ w - z &= 2(p/\kappa)^{\theta/\gamma} e^{-\theta S_r/(\gamma \mathfrak{R})} = (\bar{w} - \bar{z}) e^{\theta(S_l - S_r)/(\gamma \mathfrak{R})}, \end{aligned}$$

which yields

$$\begin{aligned} \bar{w} &= \frac{w}{2} (1 + e^{\theta(S_l - S_r)/(\gamma \mathfrak{R})}) + \frac{z}{2} (1 - e^{\theta(S_l - S_r)/(\gamma \mathfrak{R})}), \\ \bar{z} &= \frac{w}{2} (1 - e^{\theta(S_l - S_r)/(\gamma \mathfrak{R})}) + \frac{z}{2} (1 + e^{\theta(S_l - S_r)/(\gamma \mathfrak{R})}). \end{aligned}$$

The intersection between the slow wave curve in the Riemann invariants plane and the curve  $(\bar{w}, \bar{z})$  determines the values of the variables to the left of the contact discontinuity. Through this intersection we draw the line where  $w + z$  is constant, and the intersection between this line and the backward fast wave curve gives the values of the variables to the right of the contact discontinuity, cf. Figures 4 and 5.

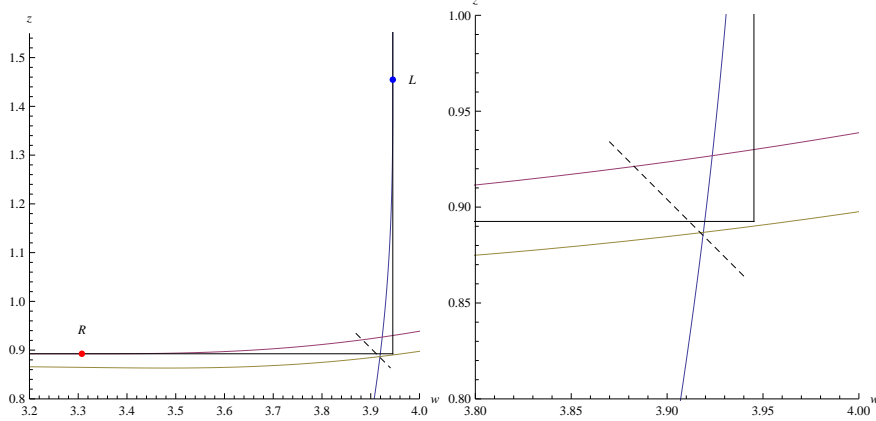


FIGURE 5. The same data is in Figure 4. Curves for the invariant region for the corresponding  $p$ -system are added (black). In addition, the dashed line is given by  $w - z$  equals a constant determined by the intersection between the yellow and blue curves. The intersection of this straight line with the red curve gives the value on the right of the contact discontinuity. The right figure is a close-up near the intersection.

The solution involves vacuum when the slow wave is a rarefaction wave that connects to a state on the vacuum line  $w = z$ ; the velocity is then given by  $u^* = u_l + \gamma^{1/2} e^{\theta S_l / \Re} \rho_l^\theta$  and  $w = z = u^*$ . Similarly, the given right state connects via a rarefaction from a vacuum state with velocity  $\tilde{u}^* = u_r - \gamma^{1/2} e^{\theta S_r / \Re} \rho_r^\theta$  and  $w = z = \tilde{u}^*$ .

### 3. PROOF OF THEOREM 1.1

**3.1. Construction of approximate solutions.** Here we provide the full proof of Theorem 1.1. We construct approximate solutions for (1.1)–(1.3) by using a Godunov-type finite difference scheme based on solving Riemann problems at each time step, updating the approximate  $S$  using the Lagrange transformation, and averaging at the end of each time step.

Before we begin the proof, let us describe the fundamentals of the construction of the approximate solution. We discretize both in space and time. Let  $h = \Delta t$ , and  $\Delta x = ch$  with  $c > 0$  to be chosen by the CFL condition

$$c > \sup_{(x,t) \in \mathbb{R} \times [0,\infty)} \left| \frac{m^h(x,t)}{\rho^h(x,t)} \pm \sqrt{p_\rho(\rho^h(x,t), S^h(x,t))} \right|,$$

which is possible as long as we can obtain an  $L^\infty$  a priori bound for

$$\frac{m^h(x,t)}{\rho^h(x,t)} \pm \sqrt{p_\rho(\rho^h(x,t), S^h(x,t))}.$$

The initial data  $\rho_0, m_0, S_0$  is approximated by step functions with jumps at  $x_{i-1/2} := (i - 1/2)\Delta x$  for  $i \in \mathbb{Z}$ . The multiple Riemann problems are solved for  $t \in [0, h)$ . At  $t = h$  a new step function is created with jumps at  $x_{i-1/2}$  (details given below), and new Riemann problems are solved. More precisely, suppose the approximate solution  $U^h = (\rho^h, m^h, \rho^h S^h)$  has been defined for  $t \leq jh$  and that  $U^h(x, jh)$  is constant for  $x \in I_i$  where

$$I_i = (x_{i-1/2}, x_{i+1/2}), \quad i \in \mathbb{Z}.$$



For  $t \in [jh, (j+1)h)$ , setting  $x_i = i\Delta x$ ,  $i \in \mathbb{Z}$ , we define  $U^h(x, t)$  by glueing together the solutions of the Riemann problems for the system (2.15)–(2.17) defined at  $[x_i, x_{i+1}] \times [jh, (j+1)h)$ , determined by the discontinuities at the points  $(x_{i+1/2}, jh)$ ,  $i \in \mathbb{Z}$ . Inductively this yields a function  $U^h$  defined on  $\mathbb{R} \times [0, \infty)$ , as long as we are able to obtain the necessary a priori bound mentioned above.

We now provide the details of the construction of the approximate solution. Assume that we have constructed the approximate solution  $U^h$  for  $x \in \mathbb{R}$  and  $t < jh$ , and have defined it at time  $t = jh$  as a piecewise constant function with jumps at  $x_{i+1/2}$  for  $i \in \mathbb{Z}$ . For  $(x, t) \in [x_i, x_{i+1}] \times [jh, (j+1)h)$ ,  $i \in \mathbb{Z}$ , let  $U^h(x, t)$  be the solution of the Riemann problem (2.15)–(2.17) as described in the previous section. Set

$$y^h(x, t) = \int_0^x \rho^h(z, t) dz - \int_0^t m^h(0, s) ds, \quad x \in \mathbb{R}, \quad t \in [jh, (j+1)h),$$

and

$$\sigma^h(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(y^h(x, t) - z)^2}{4t}} \sigma(z) dz = \tilde{\sigma}(y^h(x, t), t), \quad (x, t) \in \mathbb{R} \times [0, (j+1)h).$$

We then define<sup>2</sup>

$$(3.1) \quad \rho^h(x, (j+1)h) = \frac{1}{\Delta x} \int_{I_i} \rho^h(\tilde{x}, (j+1)h - 0) d\tilde{x},$$

$$(3.2) \quad m^h(x, (j+1)h) = \frac{1}{\Delta x} \int_{I_i} m^h(\tilde{x}, (j+1)h - 0) d\tilde{x},$$

$$(3.3) \quad S^h(x, (j+1)h) = \frac{1}{\Delta x} \int_{I_i} \sigma^h(\tilde{x}, (j+1)h - 0) d\tilde{x},$$

for  $x \in I_i$ .

**3.2. Convergence proof.** We now address the questions of the  $L^\infty$  a priori bound and convergence of  $U^h$  as  $h \rightarrow 0$ . First, we investigate the problem of obtaining an a priori  $L^\infty$  bound for the approximate solution. Let us denote

$$w^h(x, t) = w(U^h(x, t)), \quad z^h(x, t) = z(U^h(x, t)).$$

Let  $r > 0$  be such that

$$w^h(x, 0) \leq r, \quad z^h(x, 0) \geq -r, \quad x \in \mathbb{R}.$$

We assume for the moment that  $w^h, z^h$  satisfies an a priori bound of the form

$$(3.4) \quad w^h(x, t) \leq R, \quad z^h(x, t) \geq -R, \quad (x, t) \in \mathbb{R} \times [0, \infty),$$

for some constants  $R > r$ , and we will find a condition relating  $r$  and  $R$  under which (3.4) can be justified.

We first observe that if (3.4) holds, then, for any  $(x_1, t_1), (x_2, t_2) \in \mathbb{R} \times [0, \infty)$ ,

$$(3.5) \quad |y^h(x_1, t_1) - y^h(x_2, t_2)| \leq C(R)(|x_1 - x_2| + |t_1 - t_2| + h),$$

for some constant  $C(R) > 0$  depending only on  $R$ . In what follows,  $C(R)$  will always represent a positive constant depending on  $R$  that may differ from one occurrence to the next one.

<sup>2</sup>We use the standard notation  $f(x \pm 0) = \lim_{\epsilon \downarrow 0} f(x \pm \epsilon)$ .

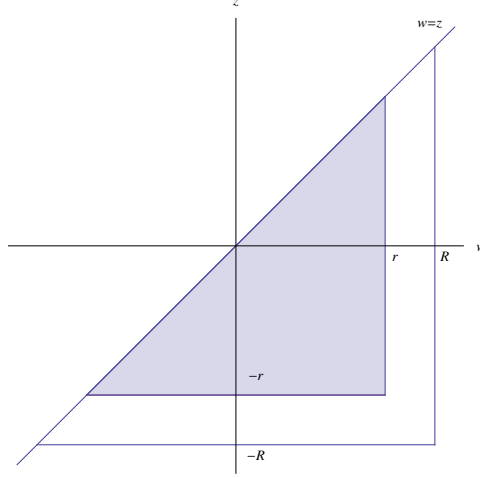


FIGURE 6. Assuming that the initial data are in the shaded region, we show the existence of an  $R$  such that the solution remains in the larger triangle. The vacuum line is  $w = z$ .

We also observe that

$$\begin{aligned}
 & |\sigma^h(x_1, t_1) - \sigma^h(x_2, t_2)| \\
 &= |\tilde{\sigma}(y^h(x_1, t_1), t_1) - \tilde{\sigma}(y^h(x_2, t_2), t_2)| \\
 &\leq |\tilde{\sigma}(y^h(x_1, t_1), t_1) - \tilde{\sigma}(y^h(x_2, t_2), t_1)| \\
 &\quad + |\tilde{\sigma}(y^h(x_2, t_2), t_1) - \tilde{\sigma}(y^h(x_2, t_2), t_2)| \\
 (3.6) \quad &\leq |y^h(x_1, t_1) - y^h(x_2, t_2)| \int_0^1 |\tilde{\sigma}_y(\tau y_2^h + (1-\tau)y_1^h, t_1)| d\tau \\
 &\quad + |t_1 - t_2| \int_0^1 |\tilde{\sigma}_t(y_2^h, \theta t_2 + (1-\theta)t_1)| d\theta \\
 &\leq C(R) \left( (|x_1 - x_2| + |t_1 - t_2| + h)e^{-t_1} + |t_1 - t_2|e^{-\min(t_1, t_2)} \right) \\
 &\leq C(R)(|x_1 - x_2| + |t_1 - t_2| + h)e^{-\min(t_1, t_2)},
 \end{aligned}$$

where we have used (1.9) and denoted  $y_i^h = y^h(x_i, t_i)$ ,  $i = 1, 2$ .

Assume inductively that

$$w^h(x, t) \leq r_j, \quad z^h(x, t) \geq -r_j, \quad (x, t) \in \mathbb{R} \times [0, jh],$$

for some constant  $r_j$ . For  $t \in [jh, (j+1)h]$  the approximate solution is defined by solving the Riemann problems given by the discontinuities at the points  $(x_{i+1/2}, jh)$ ,  $i \in \mathbb{Z}$ . Since the  $p$ -system enjoys an invariant region given in terms of  $w$  and  $z$ , the only possible increase in  $w$  beyond  $r_j$ , and, similarly, the only possible decrease in  $z$  beyond  $-r_j$ , may occur across the contact discontinuity. Here both the velocity and the pressure remain unchanged, and the sole change is in the entropy. Observe first that since the slow Riemann invariant is increasing in  $w$ , there can be no increase in the value of  $w$ . Fix  $x$  and let  $t \in [jh, (j+1)h]$ . We see from Figure 7 that the vertical line  $x$  equals a constant crosses slow or fast waves before it crosses the contact discontinuity. Let  $jh < \tilde{t} < \bar{t} < (j+1)h$  denote two times such that  $\tilde{t}$  is after the fast or slow wave, but prior to the contact discontinuity, while  $\bar{t}$  is after

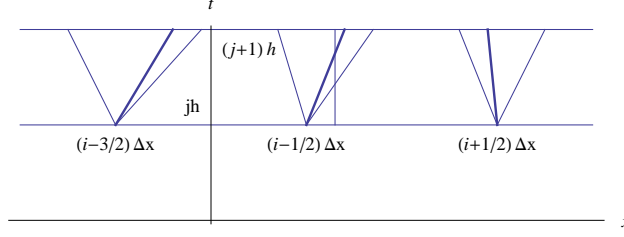


FIGURE 7. Schematic figure of the solution of the Riemann problem. Contact discontinuities are indicated by thick lines. We see that the vertical line  $x$  equals a constant first intersects a slow or a fast wave before it crosses the contact discontinuity.

the contact discontinuity. Then we find

$$\begin{aligned} z^h(x, \bar{t}) &= z^h(x, \tilde{t}) + (z^h(x, \bar{t}) - z^h(x, \tilde{t})) \\ &\geq z^h(x, \tilde{t}) - |z^h(x, \bar{t}) - z^h(x, \tilde{t})| \\ &\geq -r_j - |z^h(x, \bar{t}) - z^h(x, \tilde{t})|, \end{aligned}$$

since the solution of the  $p$ -system remains within the invariant region. Furthermore,

$$\begin{aligned} &|z^h(x, \bar{t}) - z^h(x, \tilde{t})| \\ &\leq \left| \left( \frac{p^h}{\kappa} e^{-S^h/\mathfrak{R}} \right)^{\theta/\gamma}(x, \bar{t}) - \left( \frac{p^h}{\kappa} e^{-S^h/\mathfrak{R}} \right)^{\theta/\gamma}(x, \tilde{t}) \right| \\ &\leq \left( \frac{p^h}{\kappa} e^{-S^h/\mathfrak{R}} \right)^{\theta/\gamma}(x, \bar{t}) e^{\theta S^h/(\gamma \mathfrak{R})}(x, \tilde{t}) \\ &\quad \times \left| e^{-\theta S^h/(\gamma \mathfrak{R})}(x, \tilde{t}) - e^{-\theta S^h/(\gamma \mathfrak{R})}(x, \bar{t}) \right| \\ (3.7) \quad &\leq \frac{1}{2} (w^h - z^h)(x, \bar{t}) e^{\theta S^h/(\gamma \mathfrak{R})}(x, \tilde{t}) \\ &\quad \times \left| e^{-\theta S^h/(\gamma \mathfrak{R})}(x, \tilde{t}) - e^{-\theta S^h/(\gamma \mathfrak{R})}(x, \bar{t}) \right| \\ &\leq r_j C(R)(\gamma - 1) \|S^h(\bar{t})\| \\ &= r_j C(R)(\gamma - 1) \|S^h(jh)\|, \end{aligned}$$

as we have replaced both the jump in the exponential by the corresponding jump in the exponent and estimated  $e^{\theta S^h/(\gamma \mathfrak{R})}(x, \bar{t})$  by a common constant  $C(R)$ . Next we estimate the jump in the entropy. Let  $x_1$  and  $x_2$  be two points on the left and right side of a jump, respectively, thus  $x_1 < x_{i-1/2} < x_2$ , with  $x_2 - x_1 < \Delta x$ . We obtain

$$\begin{aligned} \|S^h(jh)\| &= |S^h(x_2, jh) - S^h(x_1, jh)| \\ (3.8) \quad &\leq \frac{1}{\Delta x} \int_{I_i} |\sigma^h(\tilde{x} + \Delta x, jh) - \sigma^h(\tilde{x}, jh)| d\tilde{x} \\ &\leq C(R) h e^{-jh}, \end{aligned}$$

by (3.6).

This yields

$$(3.9) \quad z^h(x, \bar{t}) \geq -r_j(1 + C(R)(\gamma - 1)h e^{-jh}),$$

and we conclude that

$$(3.10) \quad z^h(x, t) \geq -r_j(1 + C(R)(\gamma - 1)h e^{-jh}), \quad t \in [jh, (j+1)h).$$

A similar calculation leads to

$$(3.11) \quad w^h(x, t) \leq r_j(1 + C(R)(\gamma - 1)h e^{-jh}), \quad t \in [jh, (j+1)h).$$

At  $t = (j+1)h$  we average the approximate solution as described in (3.1)–(3.3). Here we argue as follows. We first observe that the averaging of the values of  $(\rho^h(x, (j+1)h-0), m^h(x, (j+1)h-0))$  in the intervals  $I_i^{j+1} := I_i \times \{t = (j+1)h-0\}$ ,  $i \in \mathbb{Z}$ , in order to obtain the values of  $(\rho^h(x, (j+1)h), m^h(x, (j+1)h))$  in these intervals, does not affect the bounds (3.10) and (3.11). More precisely, at each such interval,  $S^h(x, (j+1)h-0)$  assumes at most 3 values, due to the possibility that two contact discontinuities, departing from  $(x_{i-1/2}, jh)$  and  $(x_{i+1/2}, jh)$ , respectively, end inside  $I_i^{j+1}$ . This means that the values of  $(\rho^h, m^h)$  in each interval  $I_i^{j+1}$  belong to the union of at most 3 regions of the form

$$R_\alpha := \{(\rho, m) : -C\rho + e^{\theta S_\alpha/\Re} \rho^{\theta+1} \leq m \leq C\rho - e^{\theta S_\alpha/\Re} \rho^{\theta+1}\}, \quad \alpha = 1, 2, 3,$$

for some constant  $C > 0$  common to all regions  $R_\alpha$ ,  $\alpha = 1, 2, 3$ . But, one easily check that  $S_1 < S_2$  implies  $R_1 \supset R_2$ , that is, the regions  $R_\alpha$ ,  $\alpha = 1, 2, 3$ , are contained in that one corresponding to  $S_* = \min\{S_1, S_2, S_3\}$ . In particular, if we define

$$S_*^h(x, (j+1)h) := \min\{S^h(\xi, (j+1)h-0) : \xi \in I_i\}, \quad \text{in } I_i^{j+1}, i \in \mathbb{Z},$$

then, from the convexity of the regions  $R_\alpha$ , we have

$$(3.12) \quad z(\rho^h, u^h, S_*^h)(x, (j+1)h) \geq -r_j(1 + C(R)(\gamma - 1)h e^{-jh}),$$

and also

$$(3.13) \quad w(\rho^h, u^h, S_*^h)(x, (j+1)h) \leq r_j(1 + C(R)(\gamma - 1)h e^{-jh}),$$

where

$$u^h(x, (j+1)h) := \begin{cases} \frac{m^h(x, (j+1)h)}{\rho^h(x, (j+1)h)}, & \text{if } \rho^h(x, (j+1)h) > 0 \\ u^h(x, (j+1)h-0), & \text{otherwise} \end{cases}$$

and we agree that the value of  $u^h(x, (j+1)h-0)$  at a vacuum interval is the mean value between its values at the extremes of the interval, which determines precisely the values of  $u^h(x, (j+1)h-0)$  for all  $x \in \mathbb{R}$ . Observe also that the case in which  $I_i^{j+1}$  is contained in a vacuum interval is trivial since  $\rho^h = m^h = 0$  in such an interval, and so the values of  $\rho^h$  and  $m^h$  do not change through averaging on  $I_i^{j+1}$ .

Now, we need to check how the bounds (3.12) and (3.13) change when we replace  $S_*^h(x, (j+1)h)$  by the values of  $S^h(x, (j+1)h)$  given by (3.3). For this, we first estimate the change in  $S^h$  from  $S^h(x, (j+1)h-0)$ , to  $S^h(x, (j+1)h)$ , given by (3.3). As already mentioned,  $S^h(x, (j+1)h-0)$  can be one of three values; either the value  $S^h(x, jh)$ , or the values of  $S$  in the neighboring intervals, that is,  $S^h(x \pm \Delta x, jh)$ . In any of the three cases, the entropy is given by a formula similar to (3.3), but with  $(j+1)h$  replaced by  $jh$ . We consider the most representative case where the value is in a neighboring interval. Thus

$$(3.14) \quad \begin{aligned} & |S^h(x, (j+1)h) - S^h(x - \Delta x, jh)| \\ & \leq \frac{1}{\Delta x} \int_{I_i} |\sigma^h(\tilde{x}, (j+1)h) - \sigma^h(\tilde{x} - \Delta x, jh)| d\tilde{x} \\ & \leq C(R)h e^{-jh}, \end{aligned}$$

again by (3.6). Since,

$$z^h(x, (j+1)h) = z(\rho^h, u^h, S^h)(x, (j+1)h), \quad w^h(x, (j+1)h) = z(\rho^h, u^h, S^h)(x, (j+1)h),$$

we conclude as above that

$$\begin{aligned}
 (3.15) \quad & z^h(x, t) \geq z(\rho^h, u^h, S_*^h)(x, (j+1)h) \\
 & \quad - |z(\rho^h, u^h, S_*^h)(x, (j+1)h) - z(\rho^h, u^h, S^h)(x, (j+1)h)| \\
 & \geq -r_j(1 + C(R)(\gamma - 1)h e^{-jh})^2 =: -r_{j+1}, \\
 & w^h(x, t) \leq w(\rho^h, u^h, S_*^h)(x, (j+1)h) \\
 & \quad + |w(\rho^h, u^h, S_*^h)(x, (j+1)h) - w(\rho^h, u^h, S^h)(x, (j+1)h)| \\
 & \leq r_j(1 + C(R)(\gamma - 1)h e^{-jh})^2 = r_{j+1}.
 \end{aligned}$$

It remains to estimate the  $r_j$ . From the inductive formula (3.15) for the  $r_j$ , we find

$$\begin{aligned}
 (3.16) \quad & r_j = r \prod_{k=1}^j (1 + C(R)(\gamma - 1)h e^{-kh})^2 \\
 & \leq r \exp \left( 2C(R)(\gamma - 1) \sum_{k=1}^j e^{-kh} h \right) \\
 & \leq r \exp \left( 2C(R)(\gamma - 1) \int_0^\infty e^{-s} ds \right) \\
 & \leq r e^{2C(R)(\gamma - 1)}.
 \end{aligned}$$

Therefore, we see from (3.16) that the condition relating  $r$  and  $R$  under which the a priori bound (3.4) holds is

$$(3.17) \quad R e^{-2(\gamma - 1)C(R)} \geq r.$$

We may easily check that  $C(R)$  may be defined as a continuous increasing function of  $R \in [0, \infty)$  such that  $C(0) = 0$  and  $C(R) \rightarrow \infty$  as  $R \rightarrow \infty$ . Hence, the left-hand side of (3.17) attains a maximum value for some  $R_* \in (0, \infty)$  and by (3.17) the initial bound  $r$  can take the largest possible value given by the left-hand side of (3.17) for  $R = R_*$ . In particular, (3.17) may be viewed as a restriction on the initial bound  $r$  which amounts a restriction on  $\|\rho_0\|_\infty$  and  $\|m_0\|_\infty$ , assuming given  $S_0$ . We also verify that the initial bound can be taken as large as we wish provided that  $\gamma - 1$  is sufficiently small.

Now we proceed to prove the compactness of the sequence of approximate solutions  $U^h$ . The proof is based on the general analysis carried out by DiPerna in [4] and we are going to apply the compactness result in [5] and its extensions in [1], [8] and [7], which together cover the whole range  $\gamma > 1$ .

Now, let  $V^h = (\rho^h, m^h)$  and  $F^h = (m^h, \rho^h(u^h)^2 + p(\rho^h, S^h))$ . For any  $\phi \in C_0^\infty(\mathbb{R}^2)$  we have

$$\begin{aligned}
 (3.18) \quad & \iint_{\mathbb{R} \times [0, \infty)} V^h \phi_t + F^h \phi_x dx dt = \sum_{j=0}^\infty \int_{jh}^{(j+1)h} \int_{\mathbb{R}} V^h \phi_t + F^h \phi_x dx dt \\
 & = \sum_{j=0}^\infty \int_{\mathbb{R}} [V^h(jh)] \phi(x, jh) dx \\
 & = \sum_{j=1}^\infty \int_{\mathbb{R}} [V^h(jh)] \phi(x, jh) dx - \int_{\mathbb{R}} V^h(x, 0) \phi(x, 0) dx,
 \end{aligned}$$

where

$$[V^h(jh)] = V^h(x, jh - 0) - V^h(x, jh + 0).$$

Further, if  $(\eta, q)$  is an arbitrary entropy pair for (1.1)–(1.2), with  $S$  constant, we have

$$(3.19) \quad \begin{aligned} \iint_{\mathbb{R} \times [0, \infty)} \eta^h \phi_t + q^h \phi_x dx dt &= \sum_{j=0}^{\infty} \int_{jh}^{(j+1)h} \int_{\mathbb{R}} \eta^h \phi_t + q^h \phi_x dx dt \\ &= - \int_{\mathbb{R}} \eta^h(x, 0) \phi(x, 0) dx + \sum_{j=1}^{\infty} \int_{\mathbb{R}} [\![\eta^h(jh)]\!] \phi(x, jh) dx + \int_0^{\infty} \mathcal{S}(\phi) dt + \int_0^{\infty} \mathcal{C}(\phi) dt, \end{aligned}$$

where, for reasons of brevity, we write  $\eta^h = \eta(V^h, S^h)$  and  $q^h = q(V^h, S^h)$ . Here

$$[\![\eta^h(jh)]\!] = \eta^h(x, jh - 0) - \eta^h(x, jh + 0),$$

and  $\mathcal{S}(\phi)$  is defined as

$$\mathcal{S}(\phi) = \sum_{\text{shocks}} (s[\![\eta^h]\!] - [\![q^h]\!]) \phi(x(t), t),$$

$$[\![\eta^h]\!] = \eta^h(x(t) - 0, t) - \eta^h(x(t) + 0, t),$$

where the sum is over all shock discontinuities  $(x(t), t)$  at time  $t$ ,  $s = x'(t)$  denoting the shock speed, while  $\mathcal{C}(\phi)$  is defined as

$$\mathcal{C}(\phi) = \sum_{\substack{\text{contact} \\ \text{discontinuities}}} (u^h[\![\eta^h]\!] - [\![q^h]\!]) \phi(x(t), t),$$

with sum running over all contact discontinuities  $(x(t), t)$  at time  $t$ , where  $u^h$  is the velocity. The latter is defined over a vacuum interval as the arithmetic mean between the velocity at the end of the 1-rarefaction wave bounding the vacuum interval on the left-hand side and the velocity at the beginning of the 2-rarefaction wave bounding the vacuum interval on the right-hand side.

We recall that if  $(\eta, q)$  is a convex entropy pair for the isentropic system (1.1)–(1.2) where  $S$  is constant, then

$$(3.20) \quad s[\![\eta^h]\!] - [\![q^h]\!] \geq 0,$$

across each shock wave. Since  $S^h$  is constant across waves of the first and third family, inequality (3.20) also holds here. Therefore, for any weak entropy pair  $(\eta, q)$ , we find that the functional

$$\int_0^{\infty} \mathcal{S}(\phi) dt$$

is a (signed) measure with locally finite total variation, as a consequence of Remark 2.1.

Concerning the functional

$$\int_0^{\infty} \mathcal{C}(\phi) dt,$$

if  $(\eta, q)$  is a smooth entropy pair, we have, in view of previous calculations,

$$|u^h[\![\eta^h(jh)]\!] - [\![q^h(jh)]\!]| \leq C_{\eta} e^{-jh} h,$$

and so

$$\left| \int_0^{\infty} \mathcal{C}(\phi) dt \right| \leq C_{\eta} \text{diam}(K) \|\phi\|_{\infty},$$

where  $K$  is any compact containing the support of  $\phi$ , which gives that this functional is also a measure with locally finite total variation.

Observe that the weak entropies may be also written as

$$\eta(\rho, u) = \rho \int_{-1}^1 g \left( \frac{m}{\rho} + z e^{(\gamma-1)S/2R} \rho^{(\gamma-1)/2} \right) (1 - z^2)_+^{\lambda} dz,$$

while a similar formula holds for  $q$ . In particular,  $\eta, q$  are Lipschitz up to vacuum if  $g$  is smooth.

We also observe that for the special entropy pair  $(\eta_*, q_*)$  we have  $\int_0^\infty \mathcal{C}(\phi) dt = 0$ . Also, for this entropy pair, for nonnegative  $\phi \in C_0^\infty(\mathbb{R}^2)$  we have

$$\begin{aligned}
 (3.21) \quad & \sum_{j=1}^{\infty} \int_{\mathbb{R}} [\eta_*^h(jh)] \phi(x, jh) dx \\
 &= \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \int_{I_i} \left( \eta_*(V^h(x, jh - 0)), S^h(x, jh + 0) \right) \\
 &\quad - \eta_*(V^h(x, jh + 0), S^h(x, jh + 0)) \phi(x, jh) dx \\
 &- \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \int_{I_i} \left( \eta_*(V^h(x, jh - 0)), S^h(x, jh + 0) \right) \\
 &\quad - \eta_*(V^h(x, jh - 0), S^h(x, jh - 0)) \phi(x, jh) dx.
 \end{aligned}$$

The first sum in the right-hand side of equation (3.21) is nonnegative for nonnegative  $\phi$ , since  $V^h(x, jh + 0)$  is the average of  $V^h(x, jh - 0)$ , in each interval  $I_i$ , and  $\eta_*$  is convex. Therefore, we get

$$\begin{aligned}
 (3.22) \quad & \sum_{j=1}^{\infty} \int_{\mathbb{R}} [\eta_*^h(jh)] \phi(x, jh) dx \\
 &\geq - \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \int_{I_i} \eta_{*S}^h(\dots) (S^h(x, jh + 0) - S^h(x, jh - 0)) \phi(x, jh) dx \\
 &\geq - \sum_{j=1}^{\infty} C e^{-jh} h \int_{\mathbb{R}} \phi(x, jh) dx,
 \end{aligned}$$

where  $\eta_{*S}^h(\dots) = \int_0^1 \eta_{*S}^h(V^h(x, jh - 0), A(\theta)) d\theta$  is the coefficient of the linear remaining term in the trivial Taylor expansion of zero order in the variable  $S$  and  $A(\theta) = (1 - \theta)S^h(x, jh - 0) + \theta S^h(x, jh + 0)$ . In particular, both the left-hand side as well as the second term of the right-hand side of (3.21) are measures of locally finite total variation. As a consequence, we may apply equality (3.21) with  $\phi$  replaced by the characteristic function of any suitably chosen rectangle  $|x| \leq L = M\Delta x$ ,  $0 \leq t \leq T = Nh$ , to find that

$$(3.23) \quad \sum_{jh \leq N} \sum_{|i\Delta x| \leq M} \int_{I_i} D_V^2 \eta_*^h(\dots) (V^h(x, jh - 0) - V^h(x, jh + 0))^2 dx \leq \text{const.},$$

for any  $M, N > 0$ , the constant depending on  $M, N$ , where  $D_V^2 \eta_*^h(\dots) = \int_0^1 (1 - \theta) D_V^2 \eta_*(B(\theta), S^h(x, jh + 0)) d\theta$  is the coefficient of the quadratic remaining term in the Taylor expansion of first order and  $B(\theta) = (1 - \theta)V^h(x, jh + 0) + \theta V^h(x, jh - 0)$ .

Since for all weak entropy  $\eta$  we have  $|D_V^2 \eta| \leq C_\eta D_V^2 \eta_*$ , for some  $C_\eta > 0$ , it follows from (3.23) that

$$(3.24) \quad \left| \sum_{jh \leq N} \sum_{|i\Delta x| \leq M} \int_{I_i} |D_V^2 \eta| (V^h(x, jh - 0) - V^h(x, jh + 0))^2 dx \right| \leq \text{const.},$$

for any  $M, N > 0$ , the constant depending on  $M, N$ .

We can then use DiPerna's method in [4] to prove the  $W_{\text{loc}}^{-1,2}$  compactness of the distributions  $\eta_t^h + q_x^h$  by decomposing the functional

$$L(\phi) = \sum_{j=1}^{\infty} \int_{\mathbb{R}} \llbracket \eta^h(jh) \rrbracket \phi(x, jh) dx$$

as

$$\begin{aligned} (3.25) \quad L(\phi) &= \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \int_{I_i} \llbracket \eta^h(jh) \rrbracket \phi(x, jh) dx \\ &= \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \left( \phi(x_i, jh) \int_{I_i} \llbracket \eta^h(jh) \rrbracket dx \right. \\ &\quad \left. + \int_{I_i} \llbracket \eta^h(jh) \rrbracket (\phi(x, jh) - \phi(x_i, jh)) dx \right) \\ &= L_1(\phi) + L_2(\phi). \end{aligned}$$

We consider the two terms separately. We have

$$L_1(\phi) = \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \phi(x_i, jh) \int_{I_i} \llbracket \eta^h(jh) \rrbracket_V + \llbracket \eta^h(jh) \rrbracket_S dx =: L_{11}(\phi) + L_{12}(\phi),$$

where, if  $\llbracket \eta(V, S) \rrbracket = \eta(V_-, S_-) - \eta(V_+, S_+)$ , we denote

$$\llbracket \eta(V, S) \rrbracket_V = \eta(V_-, S_-) - \eta(V_+, S_-), \quad \llbracket \eta(V, S) \rrbracket_S = \eta(V_+, S_-) - \eta(V_+, S_+).$$

Since  $|\llbracket \eta^h(jh) \rrbracket_S| \leq C e^{-jh} h$ , we clearly have

$$|L_{12}(\phi)| \leq C \|\phi\|_{\infty}.$$

Concerning  $L_{11}(\phi)$ , we have, cf. (3.24),

$$\begin{aligned} (3.26) \quad |L_{11}(\phi)| &\leq \left| \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \phi(i, jh) \int_{I_i} \llbracket \eta^h(jh) \rrbracket_V dx \right| \\ &= \left| \sum_{j=1}^{\infty} \sum_{i \in \mathbb{Z}} \phi(i, jh) \int_{I_i} D_V^2 \eta^h(\dots) (V^h(x, jh - 0) - V^h(x, jh + 0))^2 dx \right| \\ &\leq C \|\phi\|_{\infty}. \end{aligned}$$

Hence, we have

$$|L_1(\phi)| \leq C_1 \|\phi\|_{\infty}.$$



Next, exactly as in [4], we find, assuming that the test function  $\phi$  satisfies  $\text{supp } \phi \subseteq [-N, N] \times [-J, J]$  and keeping  $\theta > 0$  sufficiently small,

$$\begin{aligned}
 |L_2(\phi)| &\leq \sum_{|j| \leq J} \sum_{|i| \leq N} \int_{I_i} |\llbracket \eta^h(jh) \rrbracket| |\phi(x, jh) - \phi(i, jh)| dx \\
 &\leq \|\phi\|_{C^\alpha} \sum_{\substack{|j| \leq J \\ |i| \leq N}} \int_{I_i} |\llbracket \eta^h(jh) \rrbracket| \Delta x^\alpha dx \\
 &\leq \|\phi\|_{C^\alpha} \sum_{\substack{|j| \leq J \\ |i| \leq N}} \int_{I_i} \left( \frac{\Delta x^{2\alpha}}{\Delta x^\theta} + \Delta x^\theta |\llbracket \eta^h(jh) \rrbracket|^2 \right) dx \\
 &\leq \|\phi\|_{C^\alpha} \sum_{|j| \leq J} \sum_{|i| \leq N} \left( \frac{\Delta x^{2\alpha+1}}{\Delta x^\theta} + \Delta x^\theta \int_{I_i} |\llbracket \eta^h(jh) \rrbracket|^2 dx \right) \\
 &\leq \|\phi\|_{C^\alpha} \left( \frac{\Delta x^{2\alpha+1}}{\Delta x^\theta} (2J+1)(2N+1) + \Delta x^\theta \sum_{|j| \leq J} \int_{\mathbb{R}} |\llbracket \eta^h(jh) \rrbracket|^2 dx \right) \\
 &\leq \|\phi\|_{C^\alpha} \left( \frac{\Delta x^{2\alpha+1}}{\Delta x^\theta} \mathcal{O}\left(\frac{1}{\Delta x \Delta t}\right) + \Delta x^\theta \sum_{|j| \leq J} \int_{\mathbb{R}} |\llbracket \eta^h(jh) \rrbracket|^2 dx \right) \\
 &\leq C_2 \|\phi\|_{C^\alpha} \left( \frac{\Delta x^{2\alpha+1}}{\Delta x^{\theta+2}} + \Delta x^\theta \right) \\
 &\leq C_2 \|\phi\|_{C^\alpha} \Delta x^{\alpha-1/2}
 \end{aligned}$$

where  $C^\alpha$  denotes the Hölder space with seminorm

$$\|\phi\|_{C^\alpha} = \sup_{x, y \in \mathbb{R}} |\phi(x) - \phi(y)| / |x - y|^\alpha, \quad \alpha > 1/2,$$

and where  $C_2$  depends on the support of  $\phi$ . Thus

$$|L_1(\phi)| \leq C_1 \|\phi\|_\infty, \quad \text{and} \quad |L_2(\phi)| \leq C_2 (\Delta x)^\beta \|\phi\|_{C^\alpha}$$

for appropriate  $\alpha, \beta \in (0, 1)$ , for some positive constants  $C_1, C_2$  depending on  $\text{supp } \phi$ , but independent of  $\phi$ , and through the Sobolev imbedding theorem

$$L_2(\phi) \leq C_2 (\Delta x)^\beta \|\phi\|_{W^{1,q}},$$

for an appropriate  $q \in (1, 2)$  and constant depending on the support of  $\phi$ .

In this way we obtain by the usual interpolation argument that for any weak entropy pair  $(\eta, q)$  for (1.1)–(1.2) we have

$$\eta(V^h, S^h)_t + q(V^h, S^h)_x \in \{ \text{compact of } W_{\text{loc}}^{-1,2}(\mathbb{R} \times [0, \infty)) \}.$$

We can then use the compactness results in [5, 1, 8, 7] to deduce that we may extract a subsequence of  $(\rho^h, m^h, S^h)$  converging in  $L_{\text{loc}}^1(\mathbb{R} \times [0, \infty))$  to a weak solution  $(\rho(x, t), m(x, t), S(x, t))$  to (1.1)–(1.5). Also, (3.22) implies the entropy inequality (1.15), and (3.18) implies (1.10) by a calculation similar to the estimate for  $L_2(\phi)$  above.

Concerning the decay property (1.16), we prove it as follows. First, from the above discussion, we deduce that for any weak entropy pair we have

$$|\langle \eta(\rho, m, S)_t + q(\rho, m, S)_x, \phi \rangle| \leq C_1 \|\phi\|_\infty,$$

with  $C_1$  depending only on  $\text{supp } \phi$  and bounds for  $(\rho, m, S)$ . Hence, if  $U^T = (\rho^T, m^T, S^T)$  is the self-scaling sequence  $U^T(x, t) = U(Tx, Tt)$ , we see that for any

entropy pair

$$|\langle \eta(\rho^T, m^T, S^T)_t + q(\rho^T, m^T, S^T)_x, \phi \rangle| \leq C_1 \|\phi\|_\infty,$$

while from (1.15) we have, for  $0 \leq t \leq T$ ,

$$\begin{aligned} \int_{[0,L]} \eta_*(\rho, m, S)(x, t) dx &\geq \int_{[0,L]} \eta_*(\rho, m, S)(x, T) dx - C \int_t^T \int_{[0,L]} e^{-s} dx ds \\ &\geq \int_{[0,L]} \eta_*(\rho, m, S)(x, T) dx - CLe^{-t} \end{aligned}$$

Hence, we can apply the decay result in [2] to deduce (1.16), which then concludes the proof.

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